

# Improved Proof of the No-ghost Theorem for Fermion States of the Superstring<sup>1</sup>

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## Abstract

The purpose of this note is to extend the improved proof of the no-ghost theorem for the bosonic and Neveu-Schwarz dual resonance models, presented in my article Nuclear Physics B**286** (1987) 61, to cover the Ramond fermion string. As in that paper, the improvement involves the identification of an efficient basis for string state space and a self-contained proof, based on the super-Virasoro algebra, of the linear independence of the basis elements. We use our results to calculate the BRST cohomology for this system.

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# 1 Introduction

The original proof of the no-ghost theorem [1, 2] for dual resonance models (aka string theory) was substantially streamlined (and improved) some time ago [3]. The latter work focussed on the bosonic open string model, but it also sketched straightforward extensions to the Neveu-Schwarz model [4] and to the associated closed string models. However, the extension of the improved proof to the fermion (Ramond) sector [5, 6] of the superstring model [7] was not included because of complications due to the presence of fermionic zero modes. The purpose of this short note is to fill this lacuna. As a bonus, we also apply our results to the calculation of the BRST cohomology of the Ramond sector of the superstring.

The original arguments, especially those in [1], relied on an efficient basis of the state space that was an extension of one proposed in an earlier paper [8]. In particular, the linear independence of these basis states was proved in a way that, as in [8], relied on the explicit representation of the (super)-Virasoro operators in terms of normal mode oscillators. The improved arguments, offered in [3], established the required linear independence using only the algebra of the operators without regard to their representation. However, once the linear independence was established, the remainder of the proof followed exactly as in [1]. The improved proof was completely self-contained and significantly more efficient than the originals.

In the Ramond sector states generated by the super-Virasoro operators  $L_n, F_n$ , namely states of the form  $L_{-n}|\psi_1\rangle + F_{-n}|\psi_2\rangle$  for  $n > 0$ , decouple from physical amplitudes. Physical states are therefore defined as the orthogonal complement to the space of these decoupled states, in other words,

$$L_n|\text{phys}\rangle = F_n|\text{phys}\rangle = 0, \quad \text{for } n > 0 \quad (1)$$

It is convenient to call such states physical even when they do not also satisfy the on-mass-shell condition  $F_0|\text{phys}\rangle = 0$ .

The super-Virasoro algebra in  $D$  spacetime dimensions reads:

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{D}{8}n^3\delta_{n,-m} \quad (2)$$

$$[L_n, F_m] = \left(\frac{n}{2} - m\right)F_{n+m} \quad (3)$$

$$\{F_n, F_m\} = 2L_{n+m} + \frac{D}{2}n^2\delta_{n,-m} . \quad (4)$$

The special features that emerge in the critical dimension  $D = 10$  figure prominently in both the original proof and the improved versions.

As long as the spacetime dimension  $D \geq 2$ , there is a light-like direction  $k^\mu$  such that  $k \cdot p \neq 0$ , where  $p^\mu$  is the energy-momentum eigenvalue of the string state. Then one can introduce the operators  $D_n = k \cdot d_n$  and  $K_n = k \cdot a_n$  with  $k$  normalized so that  $K_0 = 1$ . Here  $a_n^\mu$  are the (bosonic) normal mode oscillators for the coordinate  $x^\mu(\sigma, \tau)$  and  $d_n^\mu$  are the (fermionic) normal modes of the worldsheet Dirac operators  $\Gamma^\mu(\sigma, \tau)$ . It will be important

in what follows that the (anti)commutators of the operators  $K_n, D_n$  among each other all vanish. We will also require their algebra with the super-Virasoro operators:

$$[L_n, K_m] = -mK_{m+n}, \quad [L_n, D_m] = -\left(m + \frac{n}{2}\right) D_{m+n} \quad (5)$$

$$[F_n, K_m] = -mD_{m+n}, \quad \{F_n, D_m\} = K_{m+n} \quad (6)$$

$$[K_n, K_m] = 0, \quad \{D_n, D_m\} = 0, \quad [K_n, D_m] = 0. \quad (7)$$

The physical states that, in addition, satisfy  $K_n|\text{phys}\rangle = 0$  for  $n > 0$  and  $D_n|\text{phys}\rangle = 0$  for  $n \geq 0$  are called the transverse states [9]. It follows from  $\{D_0, F_0\} = K_0 = 1$  that any two transverse states have vanishing inner product:

$$\langle T|T'\rangle = \langle T|\{D_0, F_0\}|T'\rangle = \langle T|(D_0F_0 + F_0D_0)|T'\rangle = 0. \quad (8)$$

Here it is understood that  $\langle T|$  is the appropriate Dirac adjoint under which the zero mode  $d_0^\mu$ , which can be represented as proportional to a Dirac matrix  $\gamma^\mu$ , is self adjoint. With this understanding, nonzero inner products require the insertion of an  $F_0$  factor:

$$\langle T|F_0|T'\rangle \neq 0. \quad (9)$$

Defining the norm with this inner product, the transverse states have nonnegative norm, relative to an overall constant factor. To see this note that they belong to the larger space of states  $|\phi\rangle$  generated by  $D_{-k}, K_{-k}, a_{-k}^i, d_{-k}^i$  where  $1 \leq i \leq D-2$  acting on a Fock vacuum  $|0\rangle^3$ . Then the inner product

$$\frac{\langle \phi|F_0|\phi'\rangle}{\langle 0|p \cdot d_0|0\rangle} \geq 0. \quad (11)$$

Then a basis for the whole space can be taken of the form [1, 3]

$$|\{f\}\{\lambda\}, \{d\}\{\kappa\}\rangle = F_0^{f_0} F_{-1}^{f_1} L_{-1}^{\lambda_1} \cdots F_{-l}^{f_l} L_{-l}^{\lambda_l} D_{-1}^{d_1} \cdots D_{-k}^{d_k} K_{-1}^{\kappa_1} \cdots K_{-k}^{\kappa_k} |T\rangle \quad (12)$$

where  $|T\rangle$  are arbitrary transverse states. Also  $\{\lambda\}$  and  $\{\kappa\}$  are bosonic partitions of two nonnegative integers. Similarly  $\{f\}$  and  $\{d\}$  are fermionic partitions of two nonnegative integers. Fermionic simply means that each  $f_i$  and  $d_i$  assumes only the values 0 or 1.

## 2 Ordering the Basis

The basis (12) is labeled by partitions of mode number. Partitions of an integer  $N$  are labeled by a sequence of nonnegative integers  $p_1, p_2, \dots, p_N$  such that  $N = \sum_{i=1}^N ip_i$ . For

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<sup>3</sup>By virtue of the zero modes  $d_0^\mu$ , the lowest mass level is degenerate. We can single one out to use as  $|0\rangle$  by, for example, forming the combinations  $d_0^k \pm id_0^{k+1}$  for  $k = 1, 3, 5, 7$  and defining  $|0\rangle$  to satisfy

$$(d_0^k + id_0^{k+1})|0\rangle = D_0|0\rangle = 0, \quad \text{and} \quad d_n^\mu|0\rangle = a_n^\mu|0\rangle = 0, \quad \text{for } n > 0. \quad (10)$$

It is, of course, understood that  $|0\rangle$  has energy momentum eigenvalue  $p^\mu$ .

bosonic partitions the  $p_i$  can assume any nonnegative integer value, whereas for fermionic partitions each  $p_i$  is either 0 or 1. A natural way to order partitions is “lexicographically” or “alphabetically”. This means that  $\{p\} < \{p'\}$  if the first nonzero entry of the sequence

$$\sum_i i(p_i - p'_i), \quad p_1 - p'_1, \quad p_2 - p'_2, \quad \dots \quad (13)$$

is positive.

The basis we employ in this paper is actually labeled by four partitions  $\{f\}, \{d\}, \{\lambda\}, \{\kappa\}$ . The first two are fermionic and the last two are bosonic. The super-Virasoro generators are controlled by  $\{f\}, \{\lambda\}$ , and it is convenient to order them jointly according to  $(\{f, \lambda\}) < (\{f', \lambda'\})$  if the first nonzero entry of the sequence

$$\sum_i i(f_i - f'_i + \lambda_i - \lambda'_i), \quad f_0 - f'_0, \quad f_1 - f'_1, \quad \lambda_1 - \lambda'_1, \quad f_2 - f'_2, \quad \dots \quad (14)$$

is positive. The operators  $K_n = k \cdot a_n$  and  $D_n = k \cdot d_n$  are controlled by the other pair of partitions  $\{d\kappa\}$ . Then the basis elements are labeled by  $(\{f\lambda\}, \{d\kappa\})$ . Then we order the entire basis according to  $(\{f\lambda\}, \{d\kappa\}) < (\{f'\lambda'\}, \{d'\kappa'\})$  if  $\{f\lambda\} < \{f'\lambda'\}$  or if  $\{f\lambda\} = \{f'\lambda'\}$  and  $\{d\kappa\} > \{d'\kappa'\}$ .

### 3 Linear Independence

To prove linear independence of the basis elements, it is convenient to define a conjugate (or “dual”) to each element (12) as follows

$$|\{f\}\{\lambda\}, \{d\}\{\kappa\}, C\rangle = F_0^{1-f_0} F_{-1}^{d_1} L_{-1}^{\kappa_1} \dots F_{-k}^{d_k} L_{-k}^{\kappa_k} D_{-1}^{f_1} \dots D_{-l}^{f_l} K_{-1}^{\lambda_1} \dots K_{-l}^{\lambda_l} |T\rangle. \quad (15)$$

As we shall see shortly the inner product of each basis element with its conjugate is not zero.

But our first task is to prove that

$$\langle \{f\}\{\lambda\}, \{d\}\{\kappa\}, C | \{f'\}\{\lambda'\}, \{d'\}\{\kappa'\} \rangle = 0, \quad \text{if } (\{f\lambda\}, \{d\kappa\}) < (\{f'\lambda'\}, \{d'\kappa'\}), \quad (16)$$

which is to say that the corresponding matrix of inner products is lower triangular. The proof is a recursive one in which we consider in turn the ways in which  $(\{f\lambda\}, \{d\kappa\}) < (\{f'\lambda'\}, \{d'\kappa'\})$ . We organize the argument as a series of steps.

**Step 1.** We first suppose  $\sum_i i(f_i - f'_i + \lambda_i - \lambda'_i) > 0$ . Then the process of moving the positive moded  $F_n, L_n$  to the right, whence they annihilate  $|T\rangle$ , leaves behind a state with negative moded  $F_{-n}, L_{-n}$  with an even smaller total mode number. Moving these to the left, whence they annihilate  $\langle T|$ , leaves behind a matrix element with at least one  $K_n$  or  $D_n$  with non-zero mode number. Since  $k$  is light-like all these operators (anti)commute with each other and any one of them kills the matrix element, establishing the claim.

**Step 2.** So next suppose  $\sum_i i(f_i - f'_i + \lambda_i - \lambda'_i) = 0$  and  $f_0 > f'_0$ , or  $f_0 = 1, f'_0 = 0$  since  $\{f\}$  is a fermionic partition. In this case we see that all factors of  $F_0$  are initially absent from the matrix element. After the matrix element is completely reduced, it will be nonzero only if a single factor of  $F_0$  is left behind. Since under the conditions of this step, no  $F_0$  is initially present, such a factor must come from a commutator such as  $[L_n, F_{-n}]$  or  $[F_n, L_{-n}]$  for some nonzero  $n$  when e.g. the positive moded super-Virasoro operators are moved to the right. But this would leave behind negative moded super-Virasoro operators of smaller total mode number than we started with, and the remaining evaluation would meet the conditions of step 1, establishing the claim.

**Step 3.** Now we suppose  $\sum_i i(f_i - f'_i + \lambda_i - \lambda'_i) = f_0 - f'_0 = 0$  and  $f_1 > f'_1$ , or  $f_1 = 1, f'_1 = 0$  since  $\{f\}$  is a fermionic partition. This means the bra contains one  $D_1$  and the ket contains no  $F_{-1}$ . Also there is precisely one factor of  $F_0$  present. Now move the  $D_1$  operator to the right until it annihilates  $|T\rangle$ . An (anti)commutator of  $D_1$  with a positive moded  $(F_n)L_n$  produces a  $(K_{n+1})D_{n+1}$ , both of which increase the mode number of the  $K_n, D_n$  factors in the bra producing the conditions of Step 1 and so yield a vanishing contribution. An anticommutator with  $F_0$  replaces  $F_0$  with  $K_1$  creating the conditions of Step 2 and a vanishing contribution. There is no  $F_{-1}$  in the ket under the conditions of this step, so the next possibility is a commutator with  $L_{-1}$  which produces a  $D_0$ . Moving this  $D_0$  to the right until it annihilates  $|T\rangle$  picks up (anti)commutators with negative moded  $(F_{-n})L_{-n}$  which reduce the total mode number of the super-Virasoro operators in the ket compared to the  $K, D$  in the bra producing the conditions of Step 1 and a vanishing contribution. Finally an (anti)commutator of  $D_1$  with  $(F_{-n})L_{-n}$  with  $n > 1$  produces a  $(D_{-(n-1)})K_{-(n-1)}$  both of which are negative moded. This procedure reduces the mode number of the super-Virasoro operators in the ket by  $n > 1$  whereas it reduces the mode number of  $D_n, K_n$  in the bra by only 1, creating the conditions of Step 1 and a vanishing contribution. Thus the claim is established.

**Step 4.** Now suppose  $\sum_i i(f_i - f'_i + \lambda_i - \lambda'_i) = f_0 - f'_0 = f_1 - f'_1 = 0$  and  $\lambda_1 > \lambda'_1$ . since  $\{\lambda\}$  is bosonic  $\lambda'_1$  is allowed to be nonzero. There is precisely one  $D_1$  in the bra and one  $F_{-1}$  in the ket. First move  $D_1$  to the right. From the reasoning of Step 3 the only nonvanishing contribution comes from the anticommutator with  $F_{-1}$  which produces  $K_0$  a nonzero number. After this reduction the matrix element contains neither  $D_1$  nor  $F_{-1}$ , and we turn to reducing the  $K_1$ 's by moving them to the right. Bearing in mind the considerations in Step 3, we see that the only nonvanishing contributions come from the commutator  $[K_1, L_{-1}]$  which produces a  $K_0$ . Since  $\lambda_1 > \lambda'_1$ , there will still be left over  $K_1$ 's in the bra after all the  $L_{-1}$ 's are removed from the ket. Moving them to the right produces no nonvanishing contributions, establishing the claim.

**Step 5.** Now suppose  $\sum_i i(f_i - f'_i + \lambda_i - \lambda'_i) = f_0 - f'_0 = f_1 - f'_1 = \lambda_1 - \lambda'_1 = 0$  and  $f_2 > f'_2$ . Start by removing the  $D_1, K_1, F_{-1}, L_{-1}$  factors from the matrix element. Then moving  $D_2$  to the right can give no nonvanishing contributions since  $f_2 = 1, f'_2 = 0$ . Proceeding in this way we eventually see that the matrix element is zero unless  $\{f\lambda\} = \{f'\lambda'\}$ .

**Step 6.** Finally we suppose  $\{f\lambda\} = \{f'\lambda'\}$  and  $\{d\kappa\} > \{d'\kappa'\}$ . Start by reducing out all the  $K_n, D_n$  in the bra against all the  $L_{-n}, F_{-n}$  in the ket. This leaves the  $L_n, F_n$  in the bra and the  $D_{-n}, K_{-n}$  in the ket. A fortiori the mode numbers are equal and there is precisely one  $F_0$  in the matrix element. So now one repeats Steps 3-5 with the role of bra and ket interchanged and  $\{d'\kappa'\}, \{d\kappa\}$  playing the roles of  $\{f\lambda\}, \{f'\lambda'\}$  respectively. The claim is then established.

As a corollary to the detailed considerations of the above argument it follows that

$$\langle \{f\}\{\lambda\}, \{d\}\{\kappa\}, C | \{f\}\{\lambda\}, \{d\}\{\kappa\} \rangle \neq 0, \quad (17)$$

In particular the transverse space is positive definite under the norm defined by the inner product (11). These conclusions follow because the only nonzero contributions that arise in the reduction process come from one of the (anti)commutators

$$[K_n, L_{-n}], \quad \{D_n, F_{-n}\}, \quad [L_n, K_{-n}], \quad \{F_n, D_{-n}\},$$

all of which are proportional to  $K_0$  a nonzero number.

We have therefore established that the matrix

$$\langle \{f\}\{\lambda\}, \{d\}\{\kappa\}, C | \{f'\}\{\lambda'\}, \{d'\}\{\kappa'\} \rangle \quad (18)$$

is triangular with nonzero diagonal entries. It follows that its determinant is not equal to zero, and hence that the states (12) are linearly independent. In other words these states form a basis of the whole state space.

## 4 Proof of No Ghost Theorem

Armed with the linearly independent basis (12) the proof of the no ghost theorem in the Ramond sector follows the original one [10]. The condition on on-shell physical states is that they are annihilated by  $F_n, L_n$  for  $n \geq 0$ . Because of the super-Virasoro algebra, it is sufficient to impose only the two conditions

$$F_0|\text{phys}\rangle = 0, \quad L_1|\text{phys}\rangle = 0. \quad (19)$$

Using the basis defined in the previous section, the first condition implies that

$$|\text{phys}\rangle = F_0|\psi\rangle \quad (20)$$

for some  $|\psi\rangle$ . This ket can be expanded in terms of the basis elements that have no  $F_0$  factor. It can be decomposed as

$$|\psi\rangle = |s\rangle + |\phi\rangle, \quad (21)$$

where  $|\phi\rangle$  contains only the basis elements that have *no* super-Virasoro factors (i.e.  $\{f\} = \{\lambda\} = 0$ , and  $|s\rangle$  contains basis elements with at least one negative moded super-Virasoro

generator. Since  $L_{-1}, F_{-1}$  generate via the algebra all other  $F_{-n}, L_{-n}$ , it follows that we can write

$$|s\rangle = L_{-1}|\psi_1\rangle + F_{-1}|\psi_2\rangle. \quad (22)$$

We now compute the action of  $L_1$ :

$$\begin{aligned} L_1 F_0(L_{-1}|\psi_1\rangle + F_{-1}|\psi_2\rangle) &= F_0(L_{-1}L_1|\psi_1\rangle + F_{-1}L_1|\psi_2\rangle) + \frac{1}{2}F_1(L_{-1}|\psi_1\rangle + F_{-1}|\psi_2\rangle) \\ &\quad + F_0\left[\left(2L_0 + \frac{D}{8}\right)|\psi_1\rangle + \frac{3}{2}F_0|\psi_2\rangle\right] \\ &= F_0|s'\rangle + |s''\rangle + \frac{3}{4}F_0|\psi_1\rangle + \left(L_0 + \frac{D}{4}\right)|\psi_2\rangle \\ &\quad + F_0\left(2L_0 + \frac{D}{8}\right)|\psi_1\rangle + \frac{3}{2}L_0|\psi_2\rangle \\ &= F_0|s'\rangle + |s''\rangle + \left(\frac{D}{4} - \frac{5}{2}\right)|\psi_2\rangle + F_0\left(\frac{D}{8} - \frac{5}{4}\right)|\psi_1\rangle \\ &\rightarrow F_0|s'\rangle + |s''\rangle, \quad \text{for } D = 10 \end{aligned} \quad (23)$$

$$L_1 F_0|\phi\rangle = F_0|\phi'\rangle + |\phi''\rangle. \quad (24)$$

The key point is that, in the critical dimension  $D = 10$ , the physical state condition

$$L_1 F_0(|s\rangle + |\phi\rangle) = 0 \quad \text{implies} \quad L_1 F_0|s\rangle = L_1 F_0|\phi\rangle = 0. \quad (25)$$

But  $L_1 F_0|s\rangle = 0$  means  $F_0|s\rangle$  is a null spurious state, and the  $L_1 F_0|\phi\rangle = 0$  implies that  $|\phi\rangle = |T\rangle$ . Hence

$$|\text{phys}\rangle = F_0|T\rangle + |\text{Null}\rangle, \quad \text{for } D = 10, \quad (26)$$

which establishes the no-ghost theorem in the fermion sector of the superstring.

## 5 BRST Cohomology

Just as was done in [3] for the bosonic string, the basis (12) can be used to give an efficient calculation of the cohomology of the BRST operator for the Ramond sector of the superstring. For other approaches to BRST cohomology see [11]. The argument below parallels that in [3] quite closely, and indeed was sketched in my review of string field theory [12] for both the Neveu-Schwarz and Ramond sectors. Here, for the sake of completeness we give a more detailed calculation for the Ramond sector.

Recall that the BRST method starts with the construction of a nilpotent Grassmann odd operator

$$Q = \sum_m c_{-m} L_m + \sum_m \gamma_{-m} F_m - \frac{1}{2} \sum_{m,n} (m-n) : c_{-m} c_{-n} b_{m+n} :$$

$$-\sum_{m,n} \gamma_{-m} \gamma_{-n} b_{m+n} + \sum_{m,n} \left( \frac{3m}{2} - n \right) : c_{-m} \gamma_{m-n} \beta_n : \quad (27)$$

$$Q^2 = 0 \quad \text{for } D = 10 \quad (28)$$

for the Ramond sector of the open superstring. Here  $b, c$  are the fermionic reparameterization ghosts and  $\beta, \gamma$  are the bosonic superghosts.

$$\{c_m, b_n\} = \delta_{m,-n}, \quad [\gamma_m, \beta_n] = \delta_{m,-n} \quad (29)$$

all other graded brackets vanishing. Total ghost number is defined as

$$G = G^b + G^\beta = c_0 b_0 + \sum_{k \neq 0} : c_{-k} b_k : + \gamma_0 \beta_0 - \sum_{k \neq 0} : \gamma_{-k} \beta_k : \quad (30)$$

where the colons denote normal ordering in the usual way.

In the BRST formalism physical states are identified with the cohomology of  $Q$ :

$$Q|\text{phys}\rangle = 0, \quad |\text{phys}\rangle \equiv |\text{phys}\rangle + Q|\Lambda\rangle \quad (31)$$

Here we will use the basis (12) augmented by ghost excitations to calculate the cohomology. The argument is significantly eased by employing the following replacements for  $F, L, c, \gamma$ :

$$F_n \rightarrow \hat{F}_n \equiv [Q, \beta_n] \quad (32)$$

$$L_n \rightarrow \hat{L}_n \equiv \{Q, b_n\} \quad (33)$$

$$c_n \rightarrow \hat{c}_n \equiv [Q, K_n], \quad n \neq 0 \quad (34)$$

$$\gamma_n \rightarrow \hat{\gamma}_n \equiv \{Q, D_n\}. \quad (35)$$

Note that since  $[Q, K_0] = 0$ , we retain  $c_0$  in its unmodified form. Except for  $c_0$ , these “dressed” operators have zero graded brackets with  $Q$ . Furthermore all the operators  $\hat{c}_n, \hat{\gamma}_n, K_n, D_n$  retain vanishing brackets with each other. In addition, the  $\hat{F}_m, \hat{L}_n$  satisfy the super-Virasoro algebra with no c-number term.

We shall identify the transverse states  $|T\rangle$  in the basis (12) with the states in the larger BRST space that satisfy

$$(b_0, D_0, \beta_0, b_n, \hat{c}_n, \beta_n, \hat{\gamma}_n, K_n, \hat{L}_n, \hat{F}_n, D_n)|T\rangle = 0, \quad n > 0. \quad (36)$$

which are equivalent to the same conditions with the hats removed. Choosing the transverse states to be annihilated by  $\beta_0$  amounts to a choice of picture (see for example [12]) and assigns 0 total ghost number to the  $|T\rangle$ . Then a basis for the larger space is obtained by applying to (12) independent monomials in  $\hat{c}_{-n}, \hat{\gamma}_{-n}$  with  $n \geq 0$ , and independent monomials in  $b_{-n}, \beta_{-n}$  for  $n > 0$ .

A final efficiency is gained by changing from the “occupation number” labelling of (12) to (an)a (anti)symmetric tensor labelling:

$$\hat{F}_{-1}^{f_1} \hat{L}_{-1}^{\lambda_1} \cdots \hat{F}_{-l}^{f_l} \hat{L}_{-l}^{\lambda_l} \leftrightarrow \hat{F}_{-[n_1]} \hat{F}_{-n_2} \cdots \hat{F}_{-n_f] \hat{L}_{-\{m_1}} \hat{L}_{-m_2} \cdots \hat{L}_{-m_l\}} \quad (37)$$



$$D_{-1}^{d_1} \cdots D_{-k}^{d_k} \leftrightarrow D_{-[n_1} D_{-n_2} \cdots D_{-n_d]} \quad (38)$$

$$K_{-1}^{\kappa_1} \cdots K_{-k}^{\kappa_k} \leftrightarrow K_{-\{n_1} K_{-n_2} \cdots K_{-n_k\}} \quad (39)$$

$$b_{-1}^{b_1} \cdots b_{-k}^{b_k} \leftrightarrow b_{-[n_1} b_{-n_2} \cdots b_{-n_b]} \quad (40)$$

$$\beta_{-1}^{\beta_1} \cdots \beta_{-k}^{\beta_k} \leftrightarrow \beta_{-\{n_1} \beta_{-n_2} \cdots \beta_{-n_\beta\}} \quad (41)$$

$$c_{-1}^{c_1} \cdots c_{-k}^{c_k} \leftrightarrow c_{-[n_1} c_{-n_2} \cdots c_{-n_c]} \quad (42)$$

$$\gamma_{-1}^{\gamma_1} \cdots \gamma_{-k}^{\gamma_k} \leftrightarrow \gamma_{-\{n_1} \gamma_{-n_2} \cdots \gamma_{-n_\gamma\}} \quad (43)$$

where  $\{ \}$ ,  $[ ]$  denote respectively complete symmetrization, complete antisymmetrization of the enclosed indices. The equivalence of these two labelling schemes is simply the known equivalence between occupation number and (anti)symmetric wave function representations of states of identical fermions or bosons. We then arrange the basis elements as follows:

$$\hat{F}_0^J \hat{F}_{-[j_1} \cdots \hat{F}_{-j_f]} \beta_{-\{k_1} \cdots \beta_{-k_\beta\}} \hat{L}_{-\{l_1} \cdots \hat{L}_{-l_l\}} b_{-[m_1} \cdots b_{-m_b]} \hat{c}_{-[n_1} \cdots \hat{c}_{-n_c]} K_{-\{p_1} \cdots K_{-p_k\}} \hat{\gamma}_{-\{q_1} \cdots \hat{\gamma}_{-q_\gamma\}} D_{-[r_1} \cdots D_{-r_d]} c_0^M \hat{\gamma}_0^N |T\rangle \quad (44)$$

As we show below the method of [3] applied to the Ramond sector, as sketched in [12], establishes that the cohomology of  $Q$  is a subspace of the space spanned by the zero mode elements of (44):

$$F_0^J c_0^M \hat{\gamma}_0^N |T\rangle, \quad N \geq 0, \quad J = 0, 1, \quad M = 0, 1 \quad (45)$$

This dramatic reduction applies without regard to the on-shell condition  $\hat{L}_0 |\text{phys}\rangle = 0$ . Writing  $|\text{phys}\rangle$  as a linear combination of (45) it is only a matter of minor algebra, given in [12], to show that the kernel of  $Q$  is limited to the two states

$$|\text{phys}\rangle_0 = F_0 |T_1\rangle, \quad \text{provided } \hat{L}_0 |\text{phys}\rangle = 0 \quad (46)$$

$$|\text{phys}\rangle_1 = (\gamma_0 + c_0 F_0) |T_2\rangle \quad (47)$$

where the subscripts indicate ghost number. Note  $Q |\text{phys}\rangle_1 = 0$  both on and off shell. Off shell we can write

$$|\text{phys}\rangle_1 = Q \frac{b_0}{\hat{L}_0} |\text{phys}\rangle_1 \quad (48)$$

showing that it is trivial. Since

$$\hat{L}_0 = \frac{p^2}{2} + \sum_{m=1}^{\infty} (\alpha_{-m} \cdot \alpha_m + d_{-m} \cdot d_m) + \sum_{m=1}^{\infty} m (c_{-m} b_m + b_{-m} c_m - \gamma_{-m} \beta_m + \beta_{-m} \gamma_m) \quad (49)$$

has a continuous spectrum in the neighborhood of 0 (since  $p^2$  is continuous), we can regard the onshell limit of  $|\text{phys}\rangle_1$  also as trivial. On the other hand  $Q |\text{phys}\rangle_0 = 0$  only on shell, so the space  $\{|\text{phys}\rangle_0\}$  is the true cohomology. This will complete the BRST demonstration of the no-ghost theorem for the Ramond sector of the superstring.

The key to proving the basis reduction (45) is the tableau identity (see *e.g.* [13], Section 7-12)

$$\underbrace{\begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline \end{array}}_{\lambda} \otimes \left\{ \begin{array}{|c|} \hline \\ \hline \end{array} \right\}^{\beta} = (\beta + 1) \left\{ \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline \end{array} \right\}^{\lambda} + \beta \left\{ \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline \end{array} \right\}^{\lambda+1} \quad (50)$$

The product of a symmetric tensor with an antisymmetric tensor produces precisely two symmetry patterns. The operators appearing in the basis can be paired off according to how they behave under bracketing with  $Q$ . Each pair has one bosonic and one fermionic member. For the  $R$  sector the pairings are

$$\hat{F}_n \leftrightarrow \beta_n, \quad \hat{L}_n \leftrightarrow b_n, \quad K_n \leftrightarrow \hat{c}_n, \quad D_n \leftrightarrow \hat{\gamma}_n \quad (51)$$

The bracket of  $Q$  with a product of two monomials, one containing the bosonic members and the other containing the fermionic members of a pairing, then shows that one of the two symmetry patterns can be gauged away, and that the requirement  $Q|\text{phys}\rangle = 0$  implies the other pattern is absent in  $|\text{phys}\rangle$ . To analyze the cohomology of  $Q$  this procedure is applied sequentially to each pair.

First we compute the brackets

$$\begin{aligned} & \left[ Q, \hat{F}_0^J \hat{F}_{-[j_1] \cdots \hat{F}_{-j_{f-1}}} \beta_{-\{k_1 \cdots \beta_{-k_{\beta+1}}\}} \right]_{\pm} \\ &= (-)^{f+J-1} \hat{F}_0^J \sum_l \hat{F}_{-[j_1] \cdots \hat{F}_{-j_{f-1}}} \beta_{-\{k_1 \cdots \hat{F}_{-k_l} \cdots \beta_{-k_{\beta+1}}\}} \end{aligned} \quad (52)$$

$$\begin{aligned} &= (-)^{f+J-1} \hat{F}_0^J \sum_l \hat{F}_{-[j_1] \cdots \hat{F}_{-j_{f-1}}} \hat{F}_{-k_l} \beta_{-\{k_1 \cdots \langle \beta_{-k_l} \rangle \cdots \beta_{-k_{\beta+1}}\}} \\ &\quad + \text{other terms with less than } J+f \text{ } \hat{F}'\text{s.} \end{aligned} \quad (53)$$

$$\begin{aligned} & \left[ Q, \hat{L}_{-\{l_1 \hat{L}_{-l_2} \cdots \hat{L}_{-l_{\ell-1}}\}} b_{-[m_1 b_{-m_2} \cdots b_{-m_{b+1}}]} \right]_{\pm} \\ &= \sum_k (-)^{k-1} \hat{L}_{-\{l_1 \cdots \hat{L}_{-l_{\ell-1}} \hat{L}_{-m_k}\}} b_{-[b_1 \cdots \langle b_{-m_k} \rangle \cdots b_{-m_{b+1}}]} \\ &\quad + \text{other terms with less than } \ell \text{ } \hat{L}'\text{s.} \end{aligned} \quad (54)$$

$$\begin{aligned} & \left[ Q, \hat{c}_{-[n_1 \cdots \hat{c}_{-n_{c-1}}]} K_{-\{p_1 \cdots K_{-p_{k+1}}\}} \right]_{\pm} \\ &= \sum_k \hat{c}_{-[n_1 \cdots \hat{c}_{-n_{c-1}} \hat{c}_{-p_k}]} K_{-\{p_1 \cdots \langle K_{-p_k} \rangle \cdots K_{-p_{k+1}}\}} \end{aligned} \quad (55)$$

$$\begin{aligned} & \left[ Q, \hat{\gamma}_{-\{q_1 \cdots \hat{\gamma}_{-q_{\gamma-1}}\}} D_{-[r_1 \cdots D_{-p_{d+1}}]} \right]_{\pm} \\ &= \sum_k (-)^{k-1} \hat{\gamma}_{-\{q_1 \cdots \hat{\gamma}_{-q_{\gamma-1}} \hat{\gamma}_{-r_k}\}} D_{-[r_1 \cdots \langle D_{-r_k} \rangle \cdots D_{-r_{d+1}}]} \end{aligned} \quad (56)$$

where  $\langle \rangle$  around an operator means it is absent. The “other terms” on the right of the first two bracket equations come from the nonzero commutators picked up in symmetrizing the

first term on the right. There are no such terms for the last two bracket equations.<sup>4</sup>

We now begin the sequential reduction of the basis elements needed to calculate the cohomology of  $Q$ , starting with the  $\hat{F}\beta$  factors. The sum on the right side of (53) can be recognized as the Young symmetrizer for the tableaux

$$f \left\{ \begin{array}{c} \overbrace{\square \square \square \square \square}^{\beta+1} \\ \square \\ \square \\ \square \\ \square \end{array} \right\}. \quad (57)$$

If we now apply (53) to a state in the subspace  $\mathcal{K}$  spanned by the states in (44) with no  $\hat{F}_{-(n>0)}$ 's or  $\beta$ 's, we learn that the linear combinations of states in (44) with  $f$   $\hat{F}_{-(n>0)}$ 's and  $\beta$   $\beta_{-n}$ 's with the symmetry (57) can be expressed as a pure gauge plus states with less than  $f$   $\hat{F}_{-(n>0)}$ 's. Since we can work in an eigenspace of  $\hat{L}_0$  there is a maximum possible value of  $f$ . Working recursively downward in  $f$  we can systematically remove all linear combinations with symmetry (57). Then the tableaux identity (50) implies that the basis elements (44) can be restricted to linear combinations for which terms with  $f$   $\hat{F}_{-(n>0)}$ 's and  $\beta$   $\beta$ 's are restricted to the symmetry pattern

$$f+1 \left\{ \begin{array}{c} \overbrace{\square \square \square \square \square}^{\beta} \\ \square \\ \square \\ \square \\ \square \end{array} \right\}. \quad (58)$$

Now consider (31) with  $|\text{phys}\rangle$  a linear combination of the states (44) where the  $\ell$ 's and  $b$ 's are symmetrized according to (58) and use (53) to conclude that

$$\begin{aligned} Q \sum_{\{j,k\}} \left( \hat{F}_{-\{j\}}^f \beta_{-\{k\}}^\beta \right)_{\{\beta, 1^f\}} \left| \phi, \{\beta, 1^f\} \right\rangle \\ = \sum \left[ \beta \left( \hat{F}_{-\{j\}}^{f+1} \beta_{-\{k\}}^{\beta-1} \right)_{\{\beta, 1^f\}} + \text{terms with less than } f+1 \hat{F}'\text{'s} \right] \left| \phi, \{\beta, 1^f\} \right\rangle \\ + \sum \left( \hat{F}_{-\{j\}}^f \beta_{-\{k\}}^\beta \right)_{\{\beta, 1^f\}} \left| \phi', -\{j\}, -\{k\} \right\rangle = 0 \end{aligned} \quad (59)$$

where the superscripts indicate the number of  $\hat{F}$ 's and  $\beta$ 's in the polynomials in parentheses. Also  $|\phi, \{\beta, 1^f\}\rangle$ ,  $|\phi', -\{j\}, -\{k\}\rangle$  belong to  $\mathcal{K}$ . In (59), we have denoted a tableau with  $\lambda$  boxes in the first row and one box in each of the next  $\beta$  rows by the symbol  $\{\lambda, 1^\beta\}$ . Now the crucial point about this equation is that the symmetry patterns of the first term in square brackets on the second line clash with those of all the terms on the last line. This means that the first line must vanish by itself. For maximal  $f$  the first term in the square bracket

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<sup>4</sup>The reordering terms on the right of the first bracket equation (53) introduce  $b$  and  $\hat{L}$  operators not present on the left. In contrast the right sides of the last three bracket equations involve only the operators appearing on the left. We put the  $\hat{F}\beta$  factors on the left in (44) so that at no point, in the sequential reduction to follow, does the reordering process introduce operators that had previously been eliminated.

must vanish by itself, which implies that  $|\phi, \{\beta, 1^f\}\rangle = 0$  for this maximal  $f$ . Then induction shows that they all vanish unless  $f = \beta = 0$ . Thus to find the cohomology of  $Q$  we may use the restricted basis

$$\hat{F}_0^J \hat{L}_{-\{l_1 \cdots l_l\}} \hat{L}_{-\{l_1 \cdots l_l\}} b_{-[m_1 \cdots m_b]} \hat{c}_{-[n_1 \cdots n_c]} K_{-\{p_1 \cdots p_k\}} \hat{\gamma}_{-\{q_1 \cdots q_\gamma\}} D_{-[r_1 \cdots r_d]} c_0^M \gamma_0^N |T\rangle \quad (60)$$

Now we repeat the argument for the  $Lb$  factors (exactly as in [3]) using (54) which shows that the symmetry pattern

$$b+1 \left\{ \begin{array}{c} \overbrace{\square \square \square \square \square}^{\ell} \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \end{array} \right\}. \quad (61)$$

can be gauged away, after which the requirement that  $Q$  annihilate  $|\text{phys}\rangle$  shows that terms with  $Lb$  factors are absent. One then moves on to the  $\hat{c}K$  factors using (55) and then the  $\hat{\gamma}D$  using (56). Thus repeating the same argument four times shows, finally, that the basis for calculating the cohomology can be reduced to

$$\hat{F}_0^J c_0^M \gamma_0^N |T\rangle = F_0^J c_0^M \gamma_0^N |T\rangle \quad (62)$$

which establishes (45) and hence completes the calculation of the cohomology of  $Q$ .

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